RECENT DEVELOPMENTS IN MULTISTATE RELIABILITY THEORY*

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Summary

We start by shortly reviewing some recent results on bounds for the availability and unavailability, to any level, in a fixed time interval for multistate monotone systems based on corresponding information on the multistate components. Secondly, we discuss some very recent ideas on how to generalize binary measures of component importance to the multistate case, especially focusing on binary type multistate coherent systems.

1. Introduction

One inherent weakness of traditional reliability theory is that the system and the components are always described just as functioning or failed. Fortunately, by now this theory is being replaced by a theory for multistate systems of multistate components. This enables one for instance in a power generation system to let the system state be the amount of power generated, or in an oil pipeline network the max flow one can get through the network. In both cases the system state is possibly measured on a discrete scale. The papers [1], [2], [3] initiating the research in this area came in the late seventies. Some more recent published papers are [4], [5], [6], [7]. A present state of the art of multistate theory, also aiming at a standardization of the terminology is given in [8].

Let $S = \{0,1,\ldots,M\}$ be the set of states of the system; the $M+1$ states representing successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Let furthermore, $C = \{1,\ldots,n\}$ be the set of components and S_i $(i=1,\ldots,n)$ the set of states of the ith component. We claim $\{0,M\} \subseteq S_i \subseteq S$. Let x_i $(i=1,\ldots,n)$ denote the state or performance level of the ith component and $\underline{x} =$

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$(x_1,\ldots,x_n)$. It is asssumed that the state, ϕ, of the system is given by the structure function $\phi = \phi(\underline{x})$. In this paper we consider two types of multistate systems:

Definition 1.1.

A system is a multistate monotone system (MMS) iff its structure ϕ satisfies

i) $\phi(\underline{x})$ is nondecreasing in each argument

ii) $\phi(\underline{0}) = 0$ and $\phi(\underline{M}) = M$ $(\underline{0}=(0,\ldots,0),\ \underline{M}=(M,\ldots,M))$.

Introduce the indicators $(j=1,\ldots,M)$ $I_j(x_i) = 1(0)$ if $x_i > j$ $(x_i<j)$, and the indicator vector $\underline{I}_j(\underline{x}) = (I_j(x_1),\ldots,I_j(x_n))$. If the states $\{0,1,\ldots,j-1\}$ correspond to the failure state when a binary approach is applied, then note that $\underline{I}_j(\underline{x})$ is the corresponding vector of binary component states.

Definition 1.2

A system is a binary type multistate monotone system (BTMMS) iff there exist binary structures ϕ_j $(j=1,\ldots,M)$, which are nondecreasing with $\phi_j(\underline{0}) = 0$, $\phi_j(\underline{1}) = 1$, such that the system's structure function ϕ satisfies $\phi(\underline{x})>j \iff \phi_j(I_j(\underline{x}))=1$ for all $j = 1,\ldots,M$ and all $\underline{x}$.

Following the binary approach mentioned above, we see that ϕ_j will uniquely determine the system's binary state from the components' binary states. Since obviously

$$\phi_j(\underline{0})=0, j=1,\ldots,M \iff \phi(\underline{0})=0, \phi_j(\underline{1})=1, j=1,\ldots,M \iff \phi(\underline{M})=M,$$

a BTMMS is always an MMS. It also follows that $\min_{1<i<n} x_i < \phi(\underline{x}) < \max_{1<i<n} x_i$. As in [7] we realize that the binary structures ϕ_j must satisfy $\phi_j(\underline{z}) > \phi_{j+1}(\underline{z})$ for all $j = 1,\ldots,M-1$, and all binary $\underline{z}$.

As an example consider a two-terminal network, for instance an oil pipeline network as the one depicted in Fig.1.

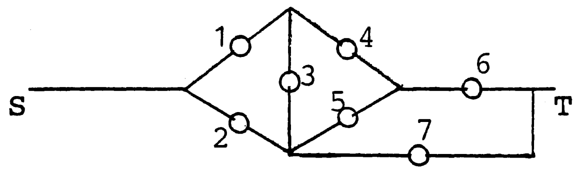

Fig.1. Example of a two-terminal network.

Assume for simplicity that the flow through the ith component (edge) is either y_i or 0, i=1,..,n. From the max-flow min-cut theorem of graph theory, the max flow capacity of the system is

$$M = M(\underline{y}) = \min_{i<j<k} \sum_{i \in K_j} y_i ,$$

where $K_1,\ldots,K_k$ are the minimal cut sets of the graph. Let now (i=1,...,n) $x_i = M(0)$ if the flow through the ith component is $y_i(0)$. Then the max flow through the system, as a function of $\underline{x}$, is given by the structure function

$$\phi(\underline{x}) = \min_{1<j<k} M^{-1} \sum_{i \in K_j} y_i x_i .$$

Note that $\phi(\underline{x})$ in general is not integervalued, which, however, can easily be dealt with by a suitable numbering of the possible values of $\phi(\underline{x})$. Furthermore, it is easily seen that the system corresponding to ϕ is a BTMMS. Finally, for each i, we may assume that there exists $(\cdot_i, \underline{x})$ such that $\phi(M_i, \underline{x}) > \phi(0_i, x)$ (otherwise we can just remove the edge from the graph). Hence all components are relevant to at least one level of the system. Extending the terminology of [8] our system is a BTMWCS (binary type multistate weakly coherent system).

2. Bounds for availabilities and unavailabilities for an MMS

We start by giving some definitions. In the following $\underline{y} < \underline{x}$ means $y_i < x_i$ for i=1,...,n, and $y_i < x_i$ for some i.

Definition 2.1. Let ϕ be the structure function of an MMS and let $j \in \{1,\ldots,M\}$. A vector $\underline{x}$ is said to be a minimal path (cut) vector to level j iff $\phi(\underline{x}) > j$ and $\phi(\underline{y}) < j$ for all $\underline{y} < \underline{x}$ ($\phi(\underline{x}) < j$ and $\phi(\underline{y}) > j$ for all $\underline{y} > \underline{x}$). The corresponding minimal path (cut) sets to level j are given by $C_\phi^j(\underline{x}) = \{i \mid x_i > 1\}$ ($D_\phi^j(\underline{x}) = \{i \mid x_i < M\}$).

Definition 2.2.

The performance process of the ith component $(i=1,\ldots,n)$ is a stochastic process $\{X_i(t),\ t\in[0,\infty)\}$, where for each fixed $t \in [0,\infty)$, $X_i(t)$ is a random variable (r.v.) which takes values in S_i. The marginal performance processes $\{X_i(t),\ t\in[0,\infty)\}$, $i=1,\ldots,n$ are independent in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\} \subset I$ the random vectors $(X_1(t_1),\ldots,X_1(t_m)),\ldots,(X_n(t_1),\ldots,X_n(t_m))$ are independent. The marginal performance process $\{X_i(t),\ t\in[0,\infty)\}$ is associated in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\} \subset I$, the r.v.'s $X_i(t_1),\ldots,X_i(t_m)$ are associated.

Definition 2.3

Let $j \in \{1,\ldots,M\}$. The availability, $h_\phi^{j(I)}$, and the unavailability, $g_\phi^{j(I)}$, to level j in the time interval I for an MMS with structure function ϕ are given by

$$h_\phi^{j(I)} = P[\phi(\underline{X}(s))\geq j\ \forall s\in I],\quad g_\phi^{j(I)} = P[\phi(\underline{X}(s))<j\ \forall s\in I].$$

Note that $h_\phi^{j(I)}+ g_\phi^{j(I)} \leq 1$, with equality for the case $I = [t,t]$. In [9] we arrive at bounds for $h_\phi^{j(I)}$ and $g_\phi^{j(I)}$, based on corresponding information on the multistate components, generalizing earlier work by the present author for the case $M = 1$. The components are assumed to be maintained and interdependent. Such bounds are of great interest when trying to predict the performance process of the system noting that exact expressions are obtainable just for trivial systems. As an example we give the following theorem. First denote the availability and unavailability to level j in I for the ith component of an MMS (C,ϕ) by $p_{i\phi}^{j(I)}$ and $q_{i\phi}^{j(I)}$ respectively, $i=1,\ldots,n;\ j=1,\ldots,M$. Introduce the $n\times M$ matrices

$$\underline{P}_\phi^{(I)} = \left(p_{i\phi}^{j(I)}\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}} \qquad \underline{Q}_\phi^{(I)} = \left(q_{i\phi}^{j(I)}\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}}$$

Theorem 2.4

Let (C,ϕ) be an MMS with the marginal performance processes of its components being mutually independent and each of them associated in I. Furthermore for $j \in \{1,\ldots,M\}$ let $\underline{y}_{k\phi}^j = (y_{1k\phi}^j,\ldots$

$.., y^j_{nk\phi})$, $k=1,\dots,n^j_\phi$ ($\underline{z}^j_{k\phi} = (z^j_{1k\phi},\dots,z^j_{nk\phi})$, $k=1,\dots,m^j_\phi$) be its minimal path (cut) vectors to level j. Define

$$\ell^{j'}_\phi(\underline{P}^{(I)}_\phi) = \max_{1\le k\le n^j_\phi} \prod_{i\in C^j_\phi(\underline{y}^j_{k\phi})} p_{i\phi}^{y^j_{ik\phi}(I)}$$

$$\bar{\ell}^{j'}_\phi(\underline{Q}^{(I)}_\phi) = \max_{1\le k\le m^j_\phi} \prod_{i\in D^j_\phi(\underline{z}^j_{k\phi})} q_{i\phi}^{z^j_{ik\phi}+1(I)}$$

$$\ell^{j*}_\phi(\underline{P}^{(I)}_\phi) = \prod_{k=1}^{m^j_\phi} \coprod_{i\in D^j_\phi(\underline{z}^j_{k\phi})} p_{i\phi}^{z^j_{ik\phi}+1(I)}$$

$$\bar{\ell}^{j*}_\phi(\underline{Q}^{(I)}_\phi) = \prod_{k=1}^{n^j_\phi} \coprod_{i\in C^j_\phi(\underline{y}^j_{k\phi})} q_{i\phi}^{y^j_{ik\phi}(I)}$$

$$L^j_\phi(\underline{P}^{(I)}_\phi) = \max\left[\ell^{j'}_\phi(\underline{P}^{(I)}_\phi), \ell^{j*}_\phi(\underline{P}^{(I)}_\phi)\right]$$

$$\bar{L}^j_\phi(\underline{Q}^{(I)}_\phi) = \max\left[\bar{\ell}^{j'}_\phi(\underline{Q}^{(I)}_\phi), \bar{\ell}^{j*}_\phi(\underline{Q}^{(I)}_\phi)\right]$$

Then

$$L^j_\phi(\underline{P}^{(I)}_\phi) \le h^{j(I)}_\phi \le 1-\bar{L}^j_\phi(\underline{Q}^{(I)}_\phi)$$

$$\bar{L}^j_\phi(\underline{Q}^{(I)}_\phi) \le g^{j(I)}_\phi \le 1-L^j_\phi(\underline{P}^{(I)}_\phi).$$

By specializing $M = 1$ and $I = [t,t]$ the bounds reduce to the familiar ones from binary theory. To apply the theorem one has to check that the marginal performance process of each component is associated in I. When these processes are Markovian, a convenient sufficient condition for this to hold, in terms of the transition intensities, is given in [10].

3. Measures of component importance for a BTMMS

A key reference to binary measures of component importance is [11]. Now we consider a BTMMS and restrict our attention to the case where the components, and hence the system, cannot be repaired. We also assume that the marginal performance processes of the components are mutually independent in $[0,\infty)$. Let $\underline{X}(t) = (X_1(t),\ldots,X_n(t))$ and assume $\underline{X}(0) = \underline{M}$. Introduce

$$P(X_i(t)\geq j) = 1-F_i^j(t) \overset{d}{=} \bar{F}_i^j(t),$$

where $F_i^j(t)$ is the distribution of lifelength in the states $\{j,\ldots,M\}$ of the ith component. Assume $F_i^j(t)$ has a density $f_i^j(t)$. For a BTMMS we now have

$$P[\phi(\underline{X}(t))\geq j] = P[\phi_j(\underline{I}_j(\underline{X}(t)))=1] = h_j(\underline{\bar{F}}^j(t)),$$

where h_j is the reliability function of the binary structure ϕ_j and $\underline{\bar{F}}^j(t) = (\bar{F}_1^j(t),\ldots,\bar{F}_n^j(t))$.

We start by generalizing the Birnbaum measure for the importance of the ith component. Let $(i=1,\ldots,n; j=1,\ldots,M)$

$$\begin{aligned} I_B^{(i,j)}(t) &= P[\text{ith component is critical at time } t \text{ for the system being in the states } \{j,\ldots,M\}] \\ &= h_j(1_i,\underline{\bar{F}}^j(t))-h_j(0_i,\underline{\bar{F}}^j(t)). \end{aligned}$$

Furthermore, let c_j be the average loss in utility when the system leaves the states $\{j,\ldots,M\}$. Assume $\sum_{j=1}^M c_j = 1$. Our suggested generalization is now

$$I_B^{(i)}(t) = \sum_{j=1}^{M} c_j\, I_B^{(i,j)}(t) \Big/ \sum_{r=1}^{n} I_B^{(r,j)}(t). \qquad (3.1)$$

Note that the definition is easily rewritten in the case of dependent components. Note also that $\Sigma_{i=1}^n\, I_B^{(i)}(t) = 1$.

An objection against this measure is that it gives the importance at fixed points of time, leaving for the analyst to determine which points are important. The two following suggestions of measures give answers to this question. The generalized Barlow-Proschan measure is given by

$$I_{B-P}^{(i)} = \sum_{j=1}^{M} c_j \, I_{B-P}^{(i,j)}, \qquad \text{where} \tag{3.2}$$

$$I_{B-P}^{(i,j)} = P[\text{ith component "causes" the system to leave the states } \{j,\ldots,M\}] = \int_0^\infty I_B^{(i,j)}(t) f_i^j(t)dt.$$

Let

Z_i^j = Reduction in remaining system time in the states $\{j,\ldots,M\}$ due to the leaving of the same states of the ith component.

Then the generalized Natvig measure is given by

$$I_N^{(i)} = \sum_{j=1}^{M} c_j \, EZ_i^j / \sum_{r=1}^{n} E(Z_r^j). \tag{3.3}$$

Different interpretations of Z_i^j can be given, corresponding to the ones given in [12] for the binary case. This leads to

$$E(Z_i^j) = \int_0^\infty I_B^{(i,j)}(t)\bar{F}_i^j(t)(-\ln\bar{F}_i^j(t))dt.$$

Hence $I_N^{(i)}$ and $I_{B-P}^{(i)}$ represent different time independent measures based firmly on the $I_B^{(i)}(t)$ measure. A preliminary comparison in [12] seems to indicate that the $I_N^{(i)}$ measure is advantageous.

Assume it does not exist binary $(\cdot_i, \underline{z})$ such that $\phi_j(1_i,\underline{z}) - \phi_j(0_i,\underline{z}) = 1$; i.e. the ith component is not relevant to level j. Due to the independence of the marginal performance processes of the components, we immediately get $I_B^{(i,j)}(t) = I_{B-P}^{(i,j)} = EZ_i^j = 0$, which is just right!

4. Measures of component importance for an MMS

Let us give some preliminary suggestions when considering an MMS. Assume that $\{X_i(t),\ t\in[0,\infty)\}$ is a continuous time Markov process. Let $(i=1,\ldots,n)$

$$P(X_i(t)=j) = P_i^j(t) \qquad , \; 0< j<M$$

$$\underline{P}(t) = [P_1^1(t),\ldots,P_1^M(t),\ldots,P_n^M(t)]$$

$$P_i^{j,k}(t,t+u) = P(X_i(t+u)=k \mid X_i(t)=j) \qquad , \; 0<k<j<M$$

$$\lambda_i^{j,k}(t) = \lim_{u\to 0} P_i^{j,k}(t,t+u)/u \qquad , \; 0<k<j<M$$

Now $P[\phi(\underline{X}(t))>j] = h_j(\underline{P}(t))$, where an exact expression for $h_j(\underline{P}(t))$ is given by inclusion-exclusion in [7]. Better methods are hopefully under way.

For the two first measures we restrict to the case where $\lambda_i^{j,k}(t) = 0$, $k < j-1$. The generalized Birnbaum measure now takes the form of (3.1) with $I_B^{(i,j)}(t)$ replaced by

$$J_B^{(i,j)}(t) = \sum_{k=1}^{M} [h_j((\underline{e}^k)_i,\underline{P}(t))-h_j((\underline{e}^{k-1})_i,\underline{P}(t))],$$

where $\underline{e}^k$ is a vector with the kth entry equal to 1 and the other entries equal to zero ($\underline{e}^0=\underline{0}$). The generalized Barlow-Proschan measure is given by (3.2) with $I_{B-P}^{(i,j)}$ replaced by

$$J_{B-P}^{(i)} = \int_0^{\infty} \sum_{k=1}^{M} [h_j((\underline{e}^k)_i,\underline{P}(t))-h_j((\underline{e}^{k-1})_i,\underline{P}(t))]P_i^k(t)\lambda_i^{k,k-1}(t)dt.$$

Finally, an attempt to generalize the Natvig measure is given by (3.3) with Z_i^j replaced by

U_i^j = Reduction in remaining system time in the states $\{j,\ldots,M\}$ due to the first downward transition of the ith component.

Let now T^j be the lifetime of a new system in the states $\{j,\ldots,M\}$, and T_i^j the lifetime of a new system in the states $\{j,\ldots,M\}$ allowing one minimal repair of the ith component when it for the first time leaves the perfect state M. As in [12] we have the interpretation $U_i^j = T_i^j-T^j$. Introduce

$$P_i^{*M}(t) = P_i^M(t)+\int_0^t P_i^M(u)\sum_{k=0}^{M-1} \lambda_i^{M,k}(u)P_i^{M,M}(u,t)du$$

$$P_i^{*\ell}(t) = \int_0^t P_i^M(u)\sum_{k=0}^{M-1} \lambda_i^{M,k}(u)P_i^{M,\ell}(u,t)du \quad , \ \ell=1,\dots,M-1$$

$$\underline{P}_i^*(t) = (P_i^{*1}(t),\dots,P_i^{*M}(t)).$$

Then

$$EU_i^j = \int_0^\infty [h_j((\underline{P}_i^*(t))_i, \underline{P}(t))-h_j(\underline{P}(t))]dt.$$

As one can see from the suggestions above, it is not obvious how to extend the ideas from the BTMMS case to the MMS case.

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